PARAMETER PLANE SINGULAR LINE SYSTEMS

by

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UNITED STATES NAVAL POSTGRADUATE SCHOOL

THESIS

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ABSTRACT

Derivation and methods for the proof of existence of singular lines on the parameter plane is presented. A method is derived for producing singular line systems. Several systems are designed and evaluated using this technique.

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I. INTRODUCTION

Methods for studying the effect of variations in parameters of a multivariable, multiloop control system have been developed by Mitrovic, Siljak, Thaler, et al (1), (2), (3).

Mitrovic's method consists of specifying as variables the two lowest order coefficients of the characteristic equation. Using the characteristic equation as a mapping function, constant zeta, omega, and sigma curves are transformed from the s-plane into the variable coefficient plane. This technique permits the adjustment of these variables so that the roots of the characteristic equation may be set at desired locations.

In 1964, Siljac extended Mitrovic's Method into the Coefficient Plane Method (2). This technique permits the utilization of arbitrary pairs of coefficients as variables.

The Parameter Plane Method is an extension of the above methods, which permits the adjustment of system parameters which are located in more than two coefficients of the characteristic equation (5), and (6). The Parameter Plane Method permits the analysis of complex systems which may be either linear or nonlinear. However, the complexity of this technique requires the computation of the curves by a computer.

In 1965, Nutting developed a computer program (PARAM-A) which calculates and plots the Parameter Plane curves for constant zeta, natural frequency and sigma (7). In general, analysis of the resulting curves results in the location of all of the roots of a characteristic equation for a given parameter (α,β) pair. However in 1966, Bowie discovered systems for which the existing Parameter Plane techniques did not locate all of the roots (8). Bowie designated these systems as singular line systems.

Section III describes the mathematical basis of singular lines and methods for proving their existence. A method is also developed for obtaining singular lines. Section IV contains examples of singular line systems. The application and analysis of singular line techniques to specific problem areas is contained in Section V. Section VI concludes this report with comments on existing singular line systems and recommendations for further investigation of singular line systems.

II. THE PARAMETER PLANE

In this section the derivation of the parameter plane equations for the linear case will be derived (9) and general rules for mapping points from the s-plane into the parameter plane will be discussed (3).

A. PARAMETER PLANE EQUATIONS

Consider the characteristic equation:

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{2-1}$$

where:

the coefficients $a_k (k=0,1,...,n)$ are real functions of two system parameters designated as α and β and s is the complex variable. Let s be written as:

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \tag{2-2}$$

where ω_n is the undamped natural frequency and ζ is the damping ratio. Letting

$$\theta = \cos^{-1}(-\zeta) \tag{2-3}$$

and substituting into the equation for s gives:

$$s = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta} \ . \tag{2-4}$$

Then s^k may be written as:

$$s^k = \omega_n^k e^{jk\theta} = \omega_n^k(\cos k\theta + j\sin k\theta) \ . \tag{2-5}$$

Defining:

$$T_k(-\zeta) \equiv \cos k\theta = \cos[k \cos^{-1}(-\zeta)]$$

$$U_k(-\zeta) \equiv \frac{\sin k\theta}{\sin\theta} = \frac{\sin[k \cos^{-1}(-\zeta)]}{\sin\theta} \quad . \qquad (2\text{-}6)$$

And substituting into the equation for s results in:

$$s^k = \omega_n^k[T_k(-\zeta) + j\sqrt{1-\zeta^2}\ U_k(-\zeta)] \quad . \qquad (2\text{-}7)$$

It may be shown that:

$$T_k(-\zeta) = (-1)^k T_k(\zeta)$$

$$U_k(-\zeta) = (-1)^k U_k(\zeta) \qquad (2\text{-}8)$$

where $T_k(\zeta)$ and $U_k(\zeta)$ are Chebyshev functions of the first and second kinds respectively and are given by the recursion formulae:

$$T_{k+1}(\zeta) - 2T_k(\zeta) + T_{k-1}(\zeta) = 0$$

$$U_{k+1}(\zeta) - 2U_k(\zeta) + U_{k-1}(\zeta) = 0 \qquad (2\text{-}9)$$

where:

$$T_0(\zeta) = 1 \qquad U_0(\zeta) = 0$$

$$T_1(\zeta) = \zeta \qquad U_1(\zeta) = -1 \quad .$$

Substituting equation (2-9) into equation (2-1) and equating the real and imaginary parts independently to zero results in:

$$\sum_{k=0}^{n} a_k \omega_n^k (-1)^E T_k(\zeta) = 0$$

$$(2\text{-}10)$$

$$\sum_{k=0}^{n} a_k \omega_n^k (-1)^{k+1} U_k(\zeta) = 0 \quad .$$

From equation (2-9) the interrelation between the Chebyshev functions is obtained:

$$T_k(\zeta) = \zeta U_k - U_{k-1}(\zeta) \ . \tag{2-11}$$

Substituting into equation (2-10):

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k a_k \omega_n^k U_{k-1}(\zeta) = 0 \\ &\sum_{k=0}^{n} (-1)^k a_k \omega_n^k U_k(\zeta) = 0 \ . \end{aligned} \tag{2-12}$$

Assume for the linear case that the coefficients a_k are linear combinations of the system parameters α and β and are of the form:

$$a_k = b_k\alpha + c_k\beta + d_k \tag{2-13}$$

where b_k, c_k, and d_k are real constants.

Substitution of equation (2-13) into equation (2-12) yields:

$$\begin{aligned} B_1\alpha + C_1\beta + D_1 = 0 \\ B_2\alpha + C_2\beta + D_2 = 0 \end{aligned} \tag{2-14}$$

where

$$B_1 = \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_{k-1}(\zeta)$$

$$B_2 = \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_k(\zeta)$$

$$C_1 = \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_{k-1}(\zeta)$$

$$C_2 = \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_k(\zeta) \tag{2-15}$$

$$D_1 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k U_{k-1}(\zeta)$$

$$D_2 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k U_k(\zeta) \quad . \tag{2-15}$$

Solution of equation (2-14) by Cramer's Rule yields the following parameter plane solution equations:

$$\alpha = \frac{C_1 D_2 - C_2 D_1}{B_1 C_2 - B_2 C_1}$$

$$\beta = \frac{B_2 D_1 - B_1 D_2}{B_1 C_2 - B_2 C_1} \quad . \tag{2-16}$$

The functional dependence of B_1, B_2, C_1, C_2, D_1, D_2 on ζ and ω_n will be omitted for simplicity in this report. A list of the algebraic form of the Chebyshev functions of the first kind $U_k(\zeta)$ is given in Appendix I. Appendix II gives a table of $U_k(\zeta)$ values for selected values of ζ.

The parameter plane, as defined by Siljak (2) is a rectangular coordinate plot with α as the abscissa and β as the ordinate. Equation (2-14) permits the mapping of points, excepting real axis points, from the s-plane into the parameter plane.

For example, ζ may be fixed $\zeta = \zeta_s$, and varying ω_n from zero to infinity results in a constant zeta curve in the parameter plane. This curve specifies the α,β pairs which will cause equation (2-1) to have a pair of complex roots with damping ratio $\zeta = \zeta_s$. Similarly fixing $\omega_n = \omega_{ns}$ and varying ζ from zero to one will produce a curve on the parameter plane which will specify the α,β pairs for which a pair of complex roots in equation (2-1) have a constant natural frequency, ω_n. It should be noted that this mapping is not conformal since a pair of complex roots on the s-plane maps into one point on the parameter plane and, as will be seen, a real axis point on the s-plane maps into a straight line on the parameter plane.

For real axis points in the s-plane note that $\zeta = 1$ and letting $\omega_n = -\sigma$, the equation for s becomes $s = -\sigma$. The characteristic equation becomes:

$$F(s) = \sum_{k=0}^{n} a_k(-\sigma)^k = 0 \quad . \tag{2-17}$$

Substitution of equation (2-13) into equation (2-17) yields:

$$F(s) = \sum_{k=0}^{n} (b_k\alpha+c_k\beta+d_k)(-\sigma)^k = 0 \quad . \tag{2-18}$$

Simplifying equation (2-18) yields:

$$\alpha B(\sigma) + \beta C(\sigma) + D(\sigma) = 0 \tag{2-19}$$

where:

$$B(\sigma) = \sum_{k=0}^{n} (-1)^k b_k \sigma^k$$

$$C(\sigma) = \sum_{k=0}^{n} (-1)^k c_k \sigma^k \tag{2-20}$$

$$D(\sigma) = \sum_{k=0}^{n} (-1)^k d_k \sigma^k \quad .$$

For a fixed value of σ, $B(\sigma)$, $C(\sigma)$, and $D(\sigma)$ are constant and equation (2-19) in the equation of a straight line on the parameter plane. The α,β pairs corresponding to this line cause equation (2-1) to have a real root at $s = -\sigma$.

B. MAPPING OF THE PARAMETER PLANE

The mapping of the s-plane into the parameter plane permits a designer to choose or adjust parameters so that characteristic equation roots lie within specified areas of the s-plane.

The rules and graphical techniques for the mapping process are discussed in detail by Siljak (2) and Thaler (13).

The major mapping rules and interpretation of the resulting curves are illustrated in the following example.

Consider a linear fourth order system with the following characteristic equation:

$$F(s) = s^4 + (\alpha+15)s^3 + (15\alpha + 50)s^2 + (50\alpha)s + 600\beta = 0 \ .$$

Figure II-1 shows the mapping of the characteristic equation in the parameter plane. Inspection of Figure II-1 results in the following observations:

1) A pair of complex conjugate points on the s-plane maps into a single point on the parameter plane.
2) A real axis point on the s-plane maps into a straight line on the parameter plane.
3) The stable area on the parameter plane, for this polynomial is defined by the positive α axis, the zeta equal zero curve, and infinity.
4) Complex roots of the polynomial, for specific values of α and β are located in the region on the parameter plane defined by the zeta equal zero and zeta equal one curves.

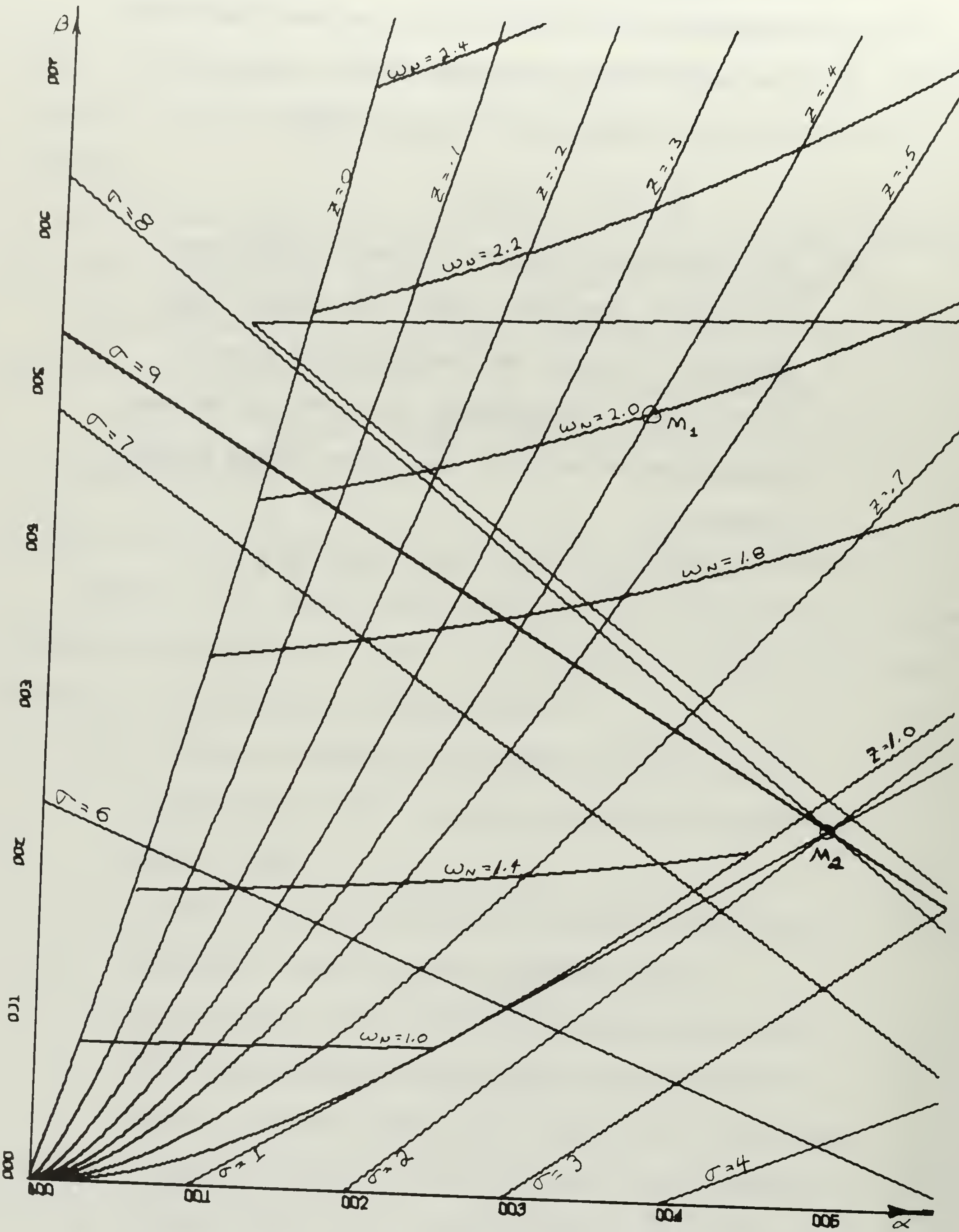
For example, point M_1 is at the intersection of the $\omega_n = 2.0$ and $\zeta = .4$ curves therefore the characteristic equation has roots at:

$$s = -(.4)(2.0) \pm j2.0\sqrt{1-(.4)^2}$$

$$= -0.8 \pm j1.83 \quad .$$

5) The remaining portion of the stable area defined by the zeta equal one curve, the positive α axis, and infinity is a real root area. For example point M_1 is at the intersection of four real root lines giving roots of:

$$s = -1.0,\ -2.0,\ -8.0,\ \text{and}\ -9.0 \quad .$$



x scale = 1.0 units/in
y scale = .10 units/in

$$s^4(\alpha+15)s^3 + (15\alpha+50)s^2 + 50\alpha s + 600\beta = 0$$

Figure II-1 Parameter Plane Example

III. SINGULAR LINES ON THE PARAMETER PLANE

The parameter plane developed by Siljak (2) and the computer program written by Nutting (7) solved for all of the roots of the characteristic equation according to then existing theory. However, Bowie (8) showed that for some systems the existing parameter plane method did not solve for all of the roots of the characteristic equation. Bowie found that these undetermined roots were actually located on a constant zeta-constant natural frequency line which has been defined as a singular line.

A. MATHEMATICAL BASIS OF SINGULAR LINES

As pointed out in Section II the solution of the parameter plane values (equation 2-16) was carried out by the application of Cramer's Rule to equation (2-14). The application of Cramer's Rule to equation (2-14) is valid only when the augmented coefficient matrix of this equation is non singular or of rank equal two. Bowie showed that when the augmented coefficient matrix of equation (2-14) is singular and the rank is equal to one, a condition exists such that β becomes a linear function of α and a singular line results. In order for the augmented coefficient matrix to be singular and of rank equal one the following conditions must prevail:

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = B_1C_2 - B_2C_1 = 0 \qquad (3\text{-}1)$$

$$\begin{vmatrix} -D_1 & C_1 \\ -D_2 & C_2 \end{vmatrix} = C_1D_2 - C_2D_1 = 0 \qquad (3\text{-}2)$$

$$\begin{vmatrix} B_1 & -D_1 \\ B_2 & -D_2 \end{vmatrix} = B_2D_1 - B_1D_2 = 0 \quad . \qquad (3\text{-}3)$$

For example, let the solution of equations (2-15) give the following values for a specific ζ, ω_n pair:

$$B_1 = 2 \qquad B_2 = 4$$

$$C_1 = 5 \qquad C_2 = 10$$

$$D_1 = 6 \qquad D_2 = 12 \quad .$$

Equations (3-1, 3-3) are satisfied and equations (2-14) become:

$$2\alpha + 5\beta + 6 = 0$$

$$4\alpha + 10\beta + 12 = 0 \quad . \qquad (3\text{-}4)$$

The resulting relationship between α and β is:

$$\beta = -\frac{2\alpha+6}{5}$$

which is a straight line on the parameter plane. This line is a locus of α,β pairs which will cause the system characteristic equation to have a pair of fixed complex roots with values of ζ and ω_n corresponding to those used in solving equation (2-15) above.

B. METHODS FOR SHOWING EXISTENCE OF SINGULAR LINES

In general one or more singular lines may be shown to exist for a particular system by the application of equations (3-1, 3-3) however, this approach requires the substitution of specific ζ, ω_n pairs into equation (2-15). Since there are an infinite number of ζ, ω_n pairs which might satisfy equations (3-1, 3-3), it is apparent that additional criteria for the possible ζ, ω_n pairs must be established.

Method I

Bowie derived the following method for finding singular lines on the parameter plane (8). Proof of the following equations will not be shown but may be found in reference (8).

The expansion of $B_1C_2 - B_2C_1$ is shown as equation (3-5) on the following page, or in compact notation

$$B_1C_2 - B_2C_1 = \sum_{\substack{i=1\\j=1}}^{n} ce_{ij} = 0 \qquad (3-6)$$

where:

$$CE = [ce_{ij}] \text{ is an } (nxn)$$

upper trangular array whose elements are:

$$\begin{cases} [b_{i-1}c_{i+j-1} - b_{i+j-1}c_{i-1}](-1)^{i-1}U_j(\zeta)\omega_n^{2i+j-2} & \text{for all } i+j < n+2 \\ 0 & \text{for all } i+j \geq n+2 \end{cases} \qquad (3-7)$$

Similarly:

$$C_1D_2 - C_2D_1 = \sum_{\substack{i=1\\j=1}}^{n} ce_{ij} = 0 \qquad (3-8)$$

$$B_1C_2-B_2C_1 =$$

$$\left|\begin{array}{l} [b_0c_1-b_1c_0]\omega_n+ \;\cdots\cdots\; +[b_{i-1}c_j-b_jc_{i-1}](-1)^{i-1}U_j(\zeta)\omega_n^j+\ldots+[b_0c_n-b_nc_0](-1)^{n-1}U_n(\zeta)\omega_n^n \\ \vdots \\ +[b_{i-1}C_i-b_ic_{i-1}]\omega_n^{2i-1}+\ldots+[b_{i-1}c_{i+j-1}-b_{i+j-1}c_{i-1}](-1)^{i+1}U_j(\zeta)\omega_n^{2i+j-2} \\ \vdots \\ +[b_{n-1}c_n-b_nc_{n-1}]\omega_n^{2n-1} \end{array}\right| \tag{3-5}$$

where:

$$CE = [ce_{ij}] \text{ is an } (nxn)$$

upper triangular array whose elements are:

$$[c_{i-1}d_{i+j-1} - c_{i+j-1}d_{i-1}](-1)^{i-1}U_j(\zeta)\omega_n^{2i+j-2} \quad \text{for all } i+j < n+2$$

$$0 \quad \text{for all } i+j \geq n+2$$

(3-9)

and

$$B_1D_2 - B_2D_1 = -\sum_{\substack{i=1 \\ j=1}}^{n} ce_{ij} = 0 \qquad (3\text{-}10)$$

where:

$$CE = [ce_{ij}] \text{ is an } (nxn)$$

upper triangular array whose elements are

$$[b_{i-1}d_{i+j-1} - b_{i+j-1}d_{i-1}](-1)^{i-1}U_j(\zeta)\omega_n^{2i+j-2} \quad \text{for all } i+j < n+2$$

$$0 \quad \text{for all } i+j \geq n+2$$

(3-11)

Equations (3-6), (3-8) and (3-10) give three polynomials in ω_n and ζ of order 2N-1 in ω_n and N-1 in ζ. For simple systems it is possible to solve for unique values of ζ and ω_n which satisfy equations (3-6), (3-8) and (3-10). For more complex systems it is necessary to set $\zeta = \zeta_s$ in one of the equations, solve for ω_n, and check the resulting values in the two remaining equations. This approach is best handled in a digital computer.

Example I

Let the system characteristic equation be:

$$F(s) = s^3 + \alpha s^2 + \beta s + 2.5\beta - 12.5 = 0 .$$

Solving equations (3-6), (3-8), and (3-10) the following is obtained:

$$B_1C_2 - B_2C_1 = 5\zeta\omega_n^2 - \omega_n^3 = 0$$

$$B_2D_1 - B_1D_2 = -\omega_n^5 + 25\zeta\omega_n^2 = 0$$

$$C_1D_2 - C_2D_1 = -2\omega_n^4\zeta + 2.5\omega_n^3(4\zeta^2-1) + 12.5\omega_n = 0 \quad .$$

Solving the above for ζ and ω_n the following unique values are obtained:

$$\zeta = \frac{1}{\sqrt{5}} \qquad \omega_n = \sqrt{5} \quad .$$

Applying these values to equation (2-14) results in:

$$5\alpha - 2.5\beta = -2.5$$

and

$$2\sqrt{5}\,\alpha - \sqrt{5}\,\beta = -\sqrt{5}$$

and the singular line relation is:

$$\beta = 2\alpha + 1 \quad .$$

Method II

The following approach to the singular line problem was developed by Cadena (10).

Assuming that a system has a singular line for some ζ, ω_n pair, there must be a pair of complex roots which are fixed for all α,β pairs in the parameter plane. Therefore, let a pair of complex roots of the characteristic equation be given by:

$$s = -\sigma \pm j\sqrt{\omega_n^2 - \sigma^2} \qquad (3\text{-}12)$$

where $\sigma = \zeta\omega_n$. After multiplication and manipulation the equation (3.12) becomes:

$$s^2 + 2\sigma + \omega_n^2 = 0 \ . \qquad (3\text{-}13)$$

If this pair of roots is divided into the characteristic equation there results a quotient which is in effect a reduced characteristic equation which has as roots the remaining roots of the original characteristic equation. Also there is, in general, a remainder in the form:

$$\sum_{k=0}^{1} s^k [f(\alpha,\beta,\sigma,\omega_n)] \ . \qquad (3\text{-}14)$$

If the complex roots, chosen as the divisor, are to be roots of the original characteristic equation, the coefficients of equation (3-14) must be independently equal to zero. Setting these coefficients equal to zero with the restriction that β is a linear function of α results in a system of polynomials which is the same as the system developed in Method I above.

<u>Example II</u>

Dividing the characteristic equation:

$$F(s) = s^3 + \alpha s^2 + \beta s + 2.5\beta - 12.5 = 0$$

(from Example I above) by:

$$s^2 + 2\sigma s + \omega_n^2$$

results in a quotient or reduced characteristic equation of:

$$s + \sigma - 2 = 0$$

and a remainder of

$$(\beta-\omega_n^2-2\sigma\alpha+4\sigma^2)s + (2.5\beta-\alpha\omega_n^2+2\sigma\omega_n^2-12.5) = 0 \ .$$

Since both terms of the remainder must be equal to zero and since β is a linear function of α the derivative of β with respect to α must be the same for both terms. From the first term:

$$\frac{d\beta}{d\alpha} = 2\sigma$$

and from the second term

$$\frac{d\beta}{d\alpha} = \frac{2\omega_n^2}{5} \ .$$

Equating the derivatives:

$$\omega_n^2 = 5\sigma \ .$$

Substitution of $\sigma = \zeta\omega_n$ yields:

$$\omega_n = 5\zeta \ .$$

Returning to the first remainder term, solving for β and substituting for β in the second remainder term results in:

$$\alpha(10\sigma-2\omega_n^2) + 5\omega_n^2 - 20\sigma^2 - 25 + 4\sigma\omega_n^2 = 0 \ .$$

Since the above equation must be satisfied for all α it must be true that the α coefficient is equal to zero and that the remaining constant terms sum to zero. From the α coefficient:

$$\omega_n^2 = 5\sigma$$

which was derived above using derivatives.

Summing the constant terms:

$$5\omega_n^2 - 20\sigma^2 - 25 + 4\sigma\omega_n^2 = 0 \ .$$

Returning to the first remainder term, solving for α in terms of β, substituting into the second remainder term and setting the sum of the constant terms equal to zero results in:

$$\omega_n^4 = 25\sigma \quad .$$

Summarizing the above relationships and letting $\sigma = \zeta\omega_n$ results in:

$$\omega_n = 5\zeta$$

$$\omega_n^3 = \frac{1}{4\zeta} [5\omega_n^2(4\zeta^2-1)+25]$$

$$\omega_n^3 = 25\zeta \quad .$$

Comparing these results with the results of Example I above shows that the singular line relationships are identical.

Solving the polynomials for ζ and ω_n results in:

$$\zeta = \frac{1}{\sqrt{5}}$$

$$\omega_n = \sqrt{5}$$

$$\sigma = 1 \quad .$$

Replacing these values in one remainder term provides the required α, β relationship for the singular line:

$$\beta = 2\alpha + 1 \quad .$$

The third root is obtained by substitution of σ into the quotient term giving:

$$s = -\alpha + 2 \quad .$$

Method III

Karmarker derived the following method for showing the existence of singular lines in the parameter plane (11).

The system characteristic equation may be written as:

$$s^n + a_n\, s^{n-1} \;.\,.\,.\,.\,.\,.\,.\; + a_1 s + a_o = 0 \tag{3-15}$$

where:

$$a_k = b_k \alpha + c_k \beta + d_k \quad . \tag{3-16}$$

Let one pair of complex roots for equation (3-15) be:

$$s = -\sigma \pm j\sqrt{\omega_n^2 - \sigma^2} \quad . \tag{3-17}$$

Now equation (3-15) may be factored into two polynomials:

$$s^2 + 2\sigma s + \omega_n^2 = 0 \tag{3-18}$$

which is obtained from equation (3-17) and:

$$s^{n-2} + x_1\, s^{n-3} \;.\,.\,.\,.\; x_{n-3} s + x_{n-2} = 0 \tag{3-19}$$

which is the reduced characteristic equation.

Multiplying equation (3-18 and (3-19) equating to the coefficient in equation (3-15) and putting the matrix notation results in:

$$[A][B] = [C]$$

where

$$A = \begin{vmatrix} b_{n-1} & c_{n-1} & -1 & 0 & 0 & \cdots & 0 \\ b_{n-2} & c_{n-2} & -2\sigma & -1 & 0 & & . \\ . & . & -\omega_n^2 & -2\sigma & -1 & & . \\ . & . & 0 & -\omega_n^2 & -2\sigma & & . \\ . & . & . & 0 & -\omega_n^2 & & . \\ . & . & . & . & 0 & & -1 \\ b_1 & c_1 & 0 & 0 & 0 & & -2\sigma \\ b_0 & c_0 & 0 & 0 & 0 & & -\omega_n^2 \end{vmatrix} \tag{3-21}$$

$$
B = \begin{vmatrix} \sigma \\ \beta \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-3} \\ x_{n-2} \end{vmatrix} \qquad C = \begin{vmatrix} 2\sigma - d_{n-1} \\ \omega_n^2 - d_{n-2} \\ -d_{n-3} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -d_1 \\ -d_0 \end{vmatrix} \qquad (3\text{-}21)
$$

For singular lines to exist β must be a linear function of α. This criterion plus the techniques of linear algebra result in the following singular line existence theorem:

Singular Line Existence Theorem

A. All determinants of order n that can be formed from the augmented matrix [A,C] are zero; (n+1) such determinants may be formed. Note that n of these form an independent set.

B. At least one of the determinants of order (n-1), that can be formed from the augmented matrix [A,C] must be non-zero. There are

$$
\begin{matrix} n+1 \\ n-1 \end{matrix} \cdot \begin{matrix} n \\ n-1 \end{matrix}
$$

such determinants.

The conditions that must exist for singular lines may be found using the existence theorem as shown in the following example.

Example III

Consider the characteristic equation used in Example II:

$$F(s) = s^3 + \alpha s^2 + \beta s + 2.5\beta - 12.5 = 0$$

$$[A] = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -2\sigma \\ 0 & 2.5 & -\omega_n^2 \end{vmatrix} \qquad [B] = \begin{vmatrix} \alpha \\ \beta \\ x \end{vmatrix} \qquad [C] = \begin{vmatrix} 2\sigma \\ \omega_n^2 \\ 12.5 \end{vmatrix}$$

The (n+1) determinants of order n which can be formed from the augmented matrix [A,C] are:

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -2\sigma \\ 0 & 2.5 & -\omega_n^2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 2\sigma \\ 0 & 1 & \omega_n^2 \\ 0 & 2.5 & 12.5 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2\sigma & -1 \\ 0 & \omega_n^2 & -2\sigma \\ 0 & 2.5 & -\omega_n^2 \end{vmatrix} \qquad \begin{vmatrix} 2\sigma & 0 & -1 \\ \omega_n^2 & 1 & -2\sigma \\ 12.5 & 2.5 & -\omega_n^2 \end{vmatrix}$$

Solving the above determinants and equating each to zero results in the following:

$$\omega_n^2 = 5\sigma$$

$$\omega_n^4 = 25\sigma$$

$$\omega_n^2 = \frac{10\sigma^2+12.5}{2\sigma+2.5}$$

$$\omega_n^2 = 5$$

Solving for σ and ω_n:

$$\sigma = 1$$

$$\omega_n = \sqrt{5}$$

It should be noted that replacing σ with $\zeta\omega_n$ in the first three equations given above results in the same set of equations as given in Examples I and II. The last equation is redundant.

Substituting for σ and ω_n in equation (3-20) and solving results in:

$$\beta = 2\alpha + 1$$

$$r_3 = x = \alpha - 2 \quad .$$

Again, this is the same result obtained in Examples I and II above.

Method IV

The following parameter plane equation and singular line theory was derived by Randolph (12).

A second derivation of the parameter plane transformation which is somewhat different than that described in Section II is described below.

Let the system characteristic equation be written as:

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{3-22}$$

where

$$a_k = b_k\alpha + c_k\beta + d_k \quad . \tag{3-23}$$

The complex variable s^k may be written as:

$$s^k = \omega_n^k e^{jk\theta} = \omega_n^k[\cos k\theta + j\sin k\theta] \tag{3-24}$$

where θ is the angle from the positive real axis to a point in the s-plane and ω_n is the natural frequency of the system.

Substituting equation (3-24 and (3-23) into equation (3-22) results in:

$$F(s) = \sum_{k=0}^{n} [b_k\alpha + c_k\beta + d_k]\omega_n^k[\cos k\theta + j\sin k\theta] = 0 \quad . \tag{3-25}$$

Letting the real and imaginary terms go independently to zero results in

$$B_1\alpha + C_1\beta + D_1 = 0$$

$$B_2\alpha + C_2\beta + D_2 = 0 \tag{3-26}$$

where

$$B_1 = \sum_{k=0}^{n} b_k\omega_n^k \cos k\theta \qquad B_2 = \sum_{k=0}^{n} b_k\omega_n^k \sin k\theta$$

$$C_1 = \sum_{k=0}^{n} c_k\omega_n^k \cos k\theta \qquad C_2 = \sum_{k=0}^{n} c_k\omega_n^k \sin k\theta$$

$$D_1 = \sum_{k=0}^{n} d_k\omega_n^k \cos k\theta \qquad D_2 = \sum_{k=0}^{n} d_k\omega_n^k \sin k\theta \quad . \tag{3-27}$$

It should be noted that the coefficients of equation (3-26) are not the same as described in Section II above.

From the theory of linear algebra it may be shown that two functions, $f_1(\alpha,\beta)$ and $f_2(\alpha,\beta)$ are linearly dependent if and only if $K_1 f_1(\alpha,\beta) + K_1 f_1(\alpha,\beta) + K_2 f_2(\alpha,\beta) = 0$ for all α,β in a domain common to f_1 and f_2 for some K_1 and K_2 not both zero.

Let:

$$f_1(\alpha,\beta) = B_1\alpha + C_1\beta + D_1$$

$$f_2(\alpha,\beta) = B_2\alpha + C_2\beta + D_2 \quad . \tag{3-28}$$

Then a necessary and sufficient condition that $f_1(\alpha,\beta)$ and $f_2(\alpha,\beta)$ be linearly dependent is that there exists a constant M such that:

$$B_2 = MB_1$$

$$C_2 = MC_1$$

$$D_2 = MC_1 \quad . \tag{3-29}$$

The conditions that must be satisfied in order for a singular line to exist, may be obtained as follows:

(a) Write equation (3-27) .

(b) Relate these equations using equations (3-29) .

(c) Solve for M, θ, and ω_n .

(d) The resulting singular line equation is obtained from equation (3-26) .

<u>Example IV</u>

Using the same characteristic equation as in previous examples:

$$F(s) = s^3 + \alpha s^2 + \beta s + 2.5\beta - 12.5 = 0$$

Equations (3-29) may be written as

$$MB_1 = M[\omega_n^2 \cos 2\theta] = \omega_n^2 \sin 2\theta = B_2$$

$$MC_1 = M[2.5+\omega_n \cos\theta] = \omega_n \sin\theta = C_2$$

$$MD_1 = M[-12.5+\omega_n^3 \cos 3\theta] = \omega_n^3 \sin 3\theta = D_2 \quad .$$

Solving for M, θ, and ω_n results in:

$$M = \frac{4}{3}$$

$$\theta = \cos^{-1}(- \frac{1}{\sqrt{5}})$$

$$\omega_n = \sqrt{5} \quad .$$

Since:

$$\theta = \cos^{-1}(-\zeta)$$

This results in a complex pair of roots at:

$$s = -1 \pm j2 \quad .$$

Returning to equation (3-26) and solving for the singular line equation results in:

$$\beta = \alpha - 2 \quad .$$

Dividing the characteristic equation by the complex roots results in the third root of the system:

$$r = \alpha - 2 \quad .$$

These results are the same as those of previous examples.

The preceding portion of this section developed methods for showing the existence of singular lines and the illustrative examples indicated procedures for finding the necessary conditions for singular lines using the existence theorems. These procedures work well for simple systems, but become complicated for the higher order systems since, in general, the order of ωn is 2 (n-1) and the order of α is n-1 in the resulting equations.

The following procedure provides a simple technique for finding singular line characteristic equations. Manipulation of the resulting characteristic equations will in general provide a singular line plant.

Let a system characteristic equation be written as:

$$F(s) = \sum_{k=0}^{n} s^k a_k = 0 \tag{3-30}$$

where:

$$a_k = \alpha b_k + \beta c_k + d_k \quad . \tag{3-31}$$

Assume that F(s) is a singular line characteristic equation for some $\zeta = \zeta_s$, $\omega_n = \omega_{ns}$. Then equation (3-30) may be rewritten in factored form as:

$$F(s) = (s^2+2\zeta_s\omega_{ns}+\omega_{ns}^2)[(s+r_1)(s+r_2)....(s+r_{n-2})] = 0 \tag{3-32}$$

In general the roots r_1, r_2, ... r_{n-2}, are functions of α and β. However for the singular line case α and β must appear in linear terms such that the order of α and β is one and $\alpha\beta$ products do not appear. Therefore only one root of $r_1 r_2 .... r_{n-2}$, may contain terms in α and β and this root must be real. The remaining roots may be fixed at any desired value. If the resulting mobile real root contains only α as a variable a_k will be a function of α. A singular line may now be obtained by substituting β as a linear function of α into one or more of the resulting a_k terms.

For example suppose that for a third order system a singular line is desired such that the roots are:

$$s = -1 \pm j2$$

$$s = \alpha - 2 \quad .$$

Substituting into equation (3-32) yields:

$$F(s) = (s^2+2s+5)(s+\alpha-2) = s^3 + \alpha s^2 + (2\alpha+1)s + 5\alpha - 10 = 0 \quad . \tag{3-33}$$

Let $\beta = 2\alpha + 1$ and substitution into equation (3-33) results in:

$$F(s) = s^3 + \alpha s^2 + \beta s + 2.5\beta - 12.5 = 0 \quad . \tag{3-34}$$

Equation (3-34) is a singular line characteristic equation for $\beta = 2\alpha + 1$ as shown in previous examples.

The procedure indicated above results in a unique singular line for a given characteristic equation. A more general case is examined by Bowie (8) in which there are, apparently, an infinite number of singular lines for a given characteristic equation. The system examined is a complex cross-coupled type with powers of s to the sixth power occurring in the characteristic equation. There does not appear to be a tractable method for producing similar systems at the present time.

IV. EXAMPLES OF SYSTEMS WITH SINGULAR LINES

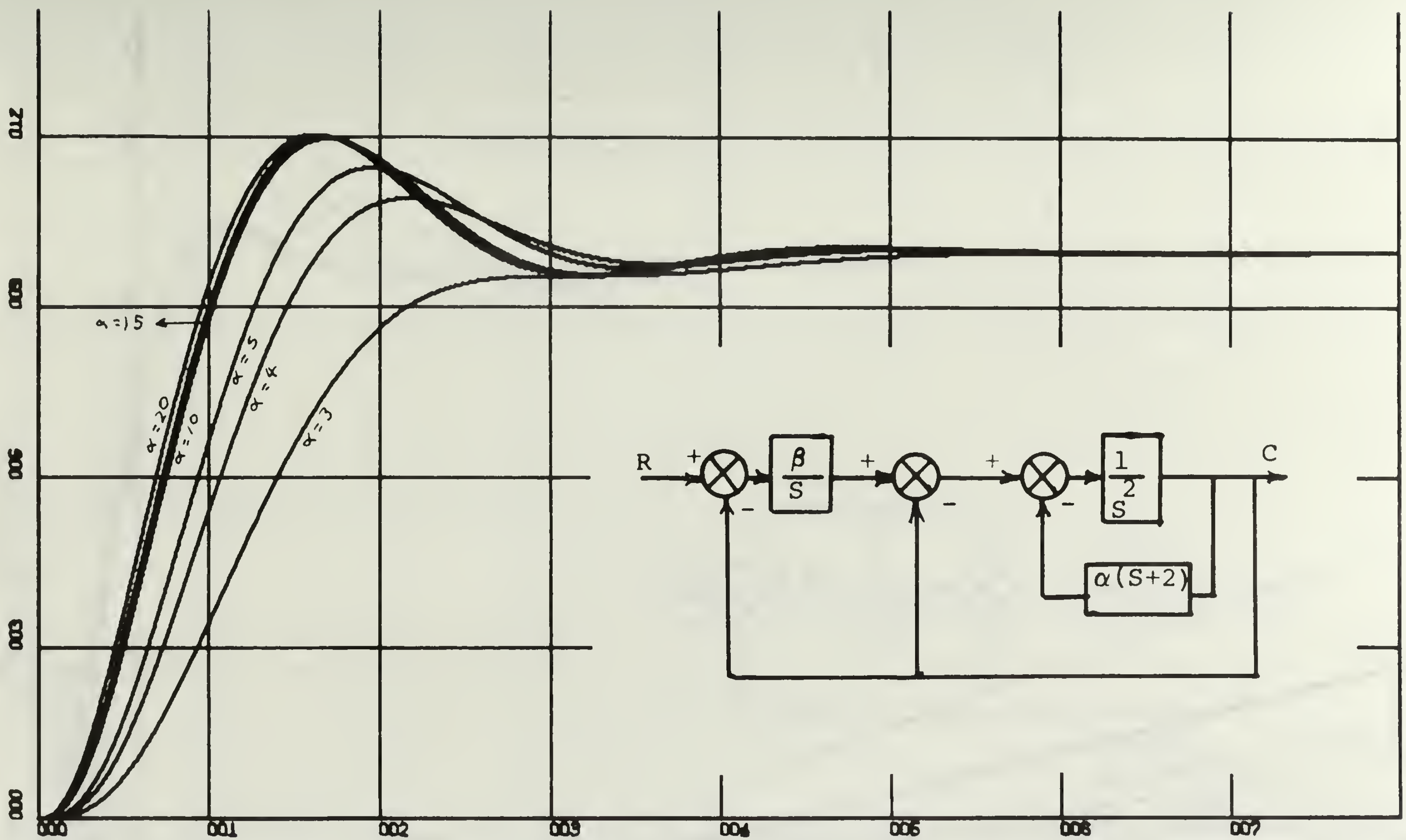
This section contains block diagrams, characteristic curves, and step response characteristics for systems which have singular lines. The systems are grouped by the nature of their characteristic equations.

A. SYSTEMS WITH ONE SINGULAR LINE AND ONE MOBILE ROOT

Ex.	Characteristic Equation	Roots	Singular Line	Figs.
1	$s^3+\alpha s^2+(2\alpha+1)s+\beta=0$	$s=-1\pm j2$ $s=-(\alpha-2)$	$\beta=5(\alpha-2)$	IV-1 IV-2
2	$s^3+\alpha s^2+\beta s+2.5\beta-100=0$	$s=-1\pm j2$ $s=-(\alpha-2)$	$\beta=2\alpha+1$	IV-3 IV-4
3	$s^3+\alpha s^2+(4\beta+5)s+\beta=0$	$s=-1\pm j2$ $s=-(\alpha-2)$	$\beta=5(\alpha-2)$	IV-5 IV-6
4	$s^4+(\alpha+10)s^3+(12\alpha+1)s^2$ $+ 25\alpha s+10\beta=0$	$s=-1\pm j2$ $s=-(\alpha-2)$ $s=-10$	$\beta=5(\alpha-2)$	IV-7 IV-8
5	$s^4+(\alpha+10)s^3+(12\alpha+1)s^2$ $+ \beta s+2\beta-100=0$	$s=-1\pm j2$ $s=-(\alpha-2)$ $s=-10$	$\beta=25\alpha$	IV-9 IV-10
6	$s^4+(\alpha+10)s^3+(12\alpha+1)s^2$ $+ (5\beta+50)s+10\beta=0$	$s=-1\pm j2$ $s=-(\alpha-2)$ $s=-10$	$\beta=5(\alpha-2)$	IV-11 IV-12

As α is increased the mobile root moves further from the origin of the s-plane and the effect of this root becomes negligible for values of α greater than 20. This trend is shown by the step response characteristics for the respective systems.

The movement of the mobile root along the real axis of the s-plane maps along the singular line in the parameter plane; this is shown by substituting $x = -(\alpha-2)$ into the system characteristic equation. In the fourth order systems, the fixed real root, $s = -10$, also maps along the singular line.



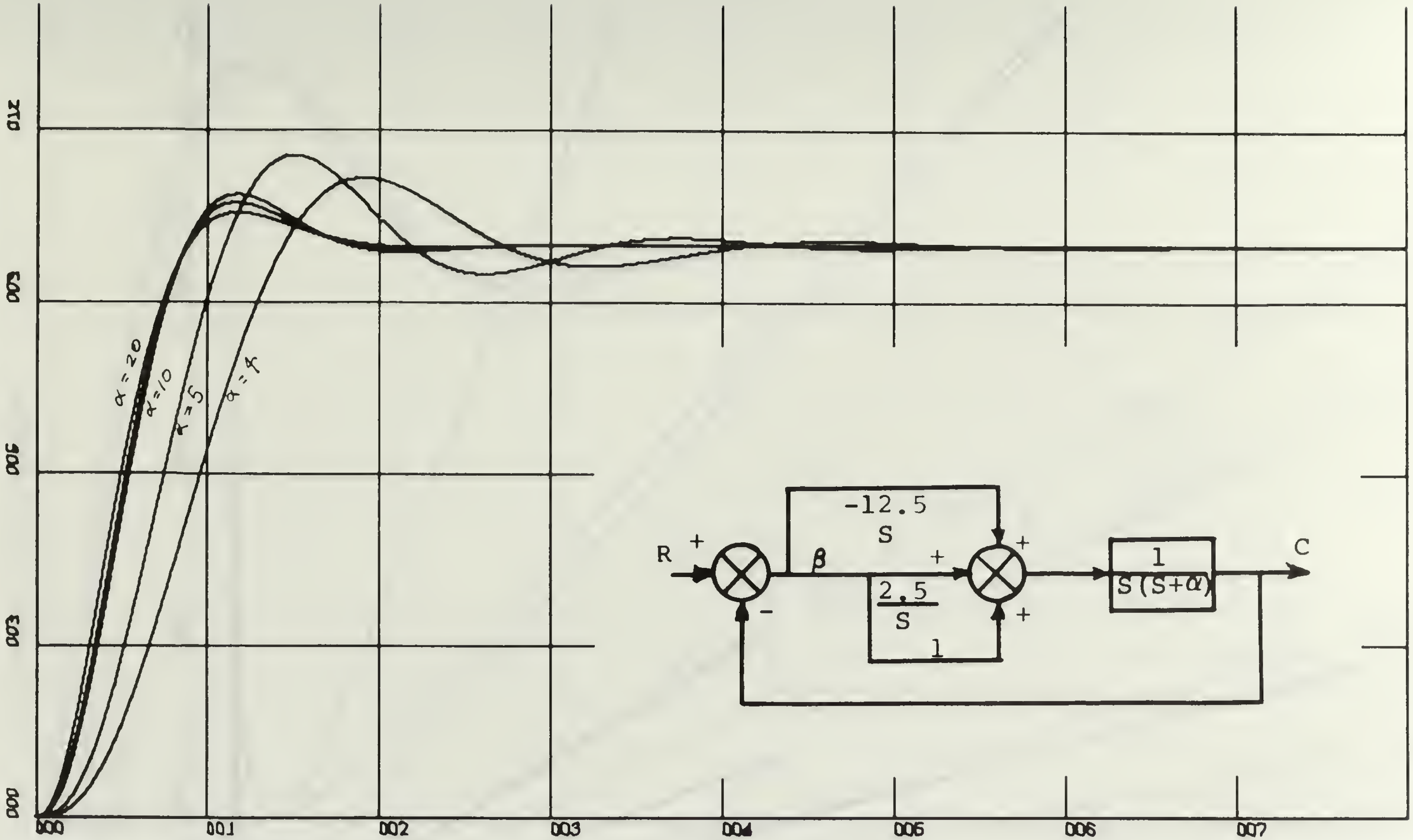
x scale = 1.00 E.00 units/in
y scale = 3.00 E-01 units/in

Figure IV-1 Step Response Curves - Example 1

x scale = 20 units/in
y scale = 20 units/in

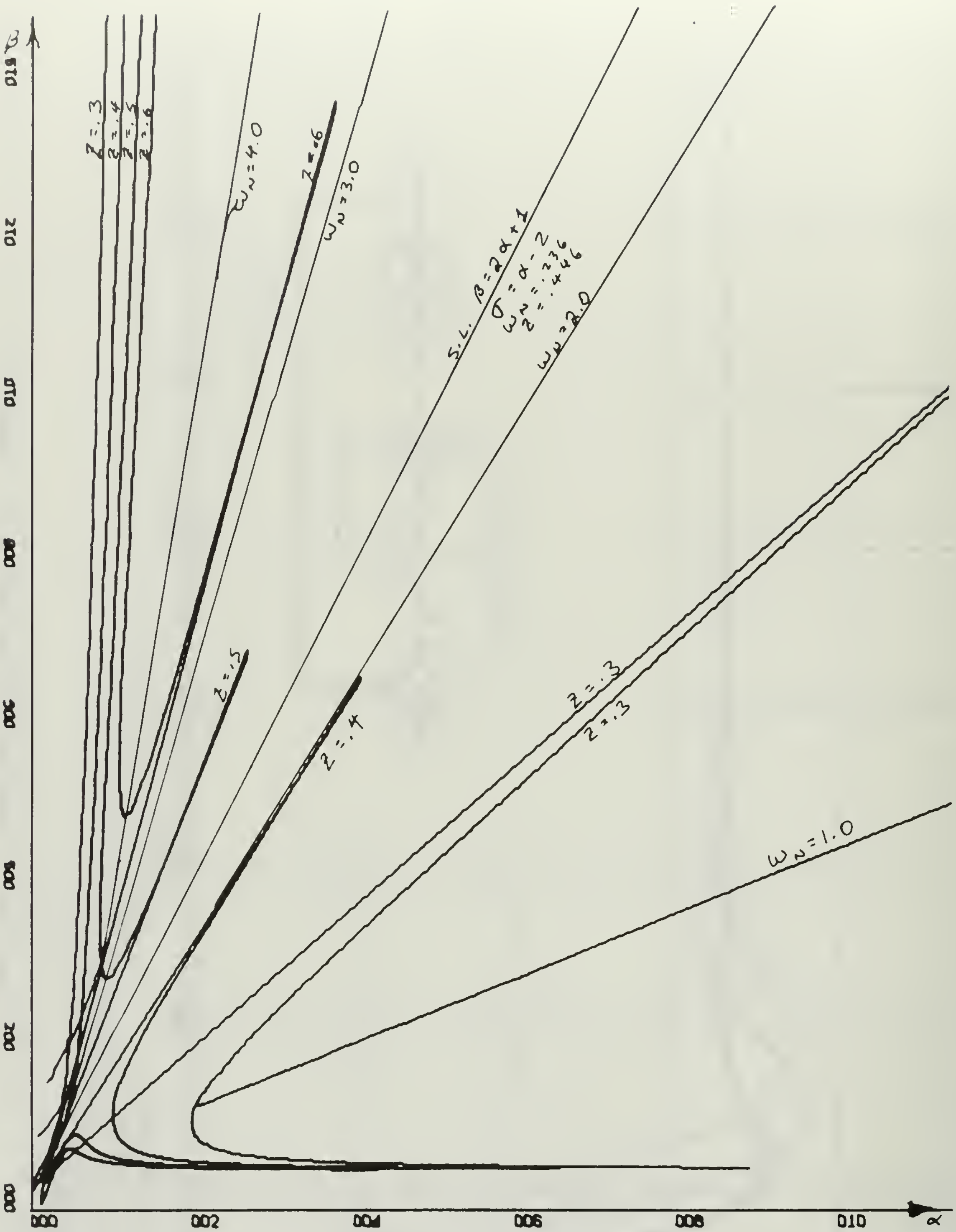
$$s^3+\alpha s^2+(2\alpha+1)s+\beta=0$$

Figure IV-2 Parameter Plane Curves - Example 1



x scale = 1.0 units/in
y scale = .3 units/in

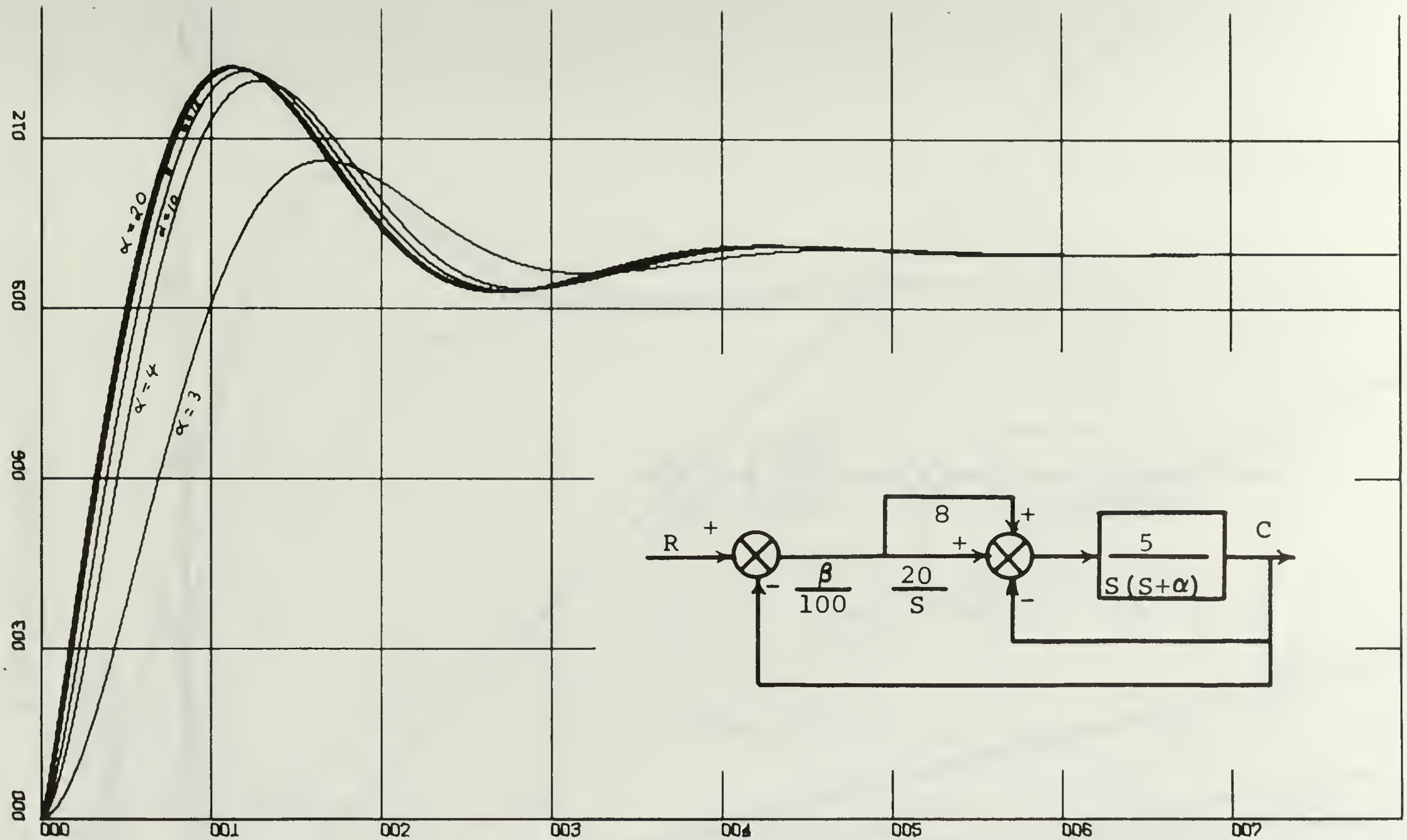
Figure IV-3 Step Response Curves - Example 2



x scale = 20 units/in
y scale = 20 units/in

$$s^3+\alpha s^2+\beta s+2.5\beta-100=0$$

Figure IV-4 Parameter Plane Curves - Example 2



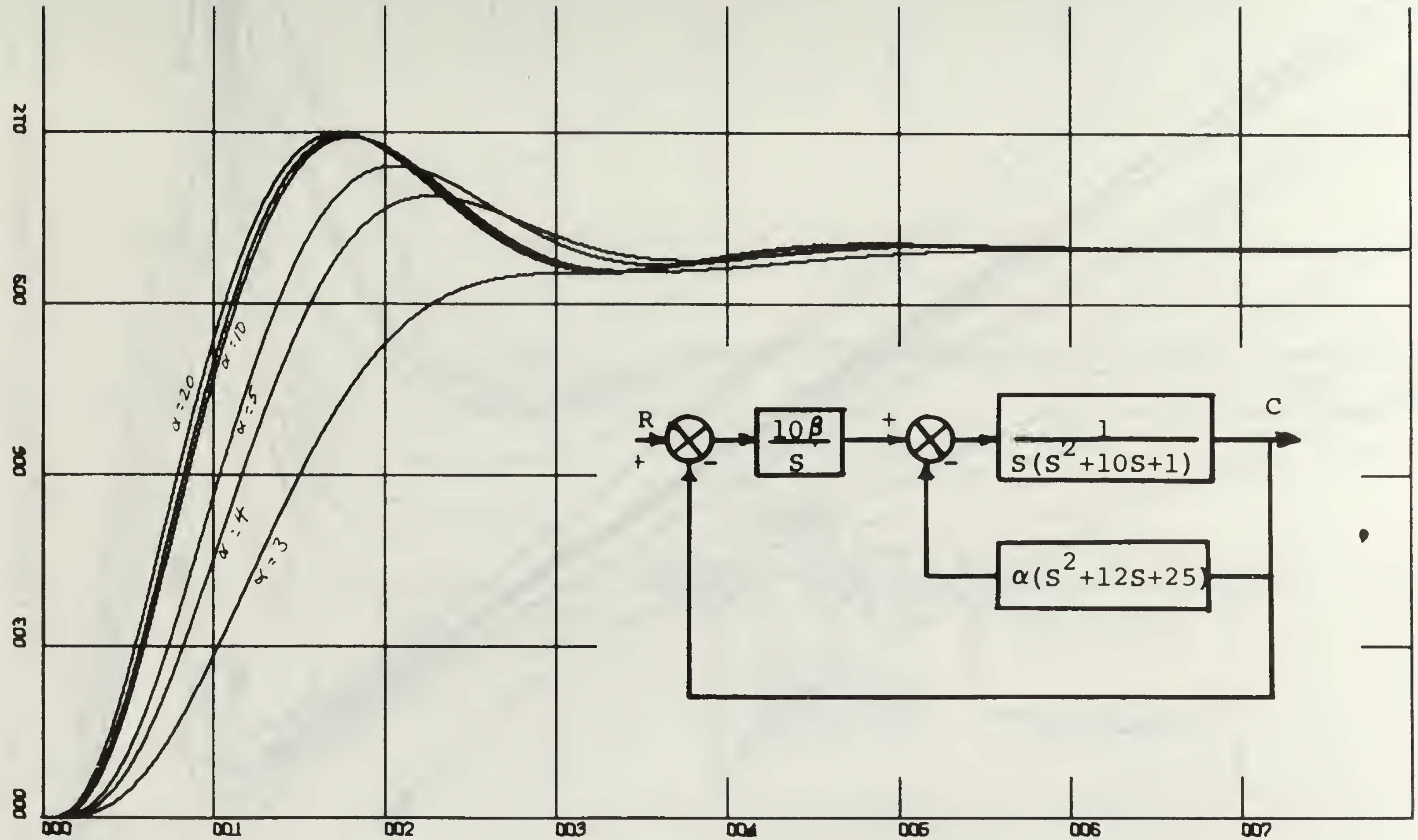
x scale = 1.00 units/in
y scale = 3.00 E-01 units/in

Figure IV-5 Step Response Curves - Example 3

x scale = 20 units/in
y scale = 20 units/in

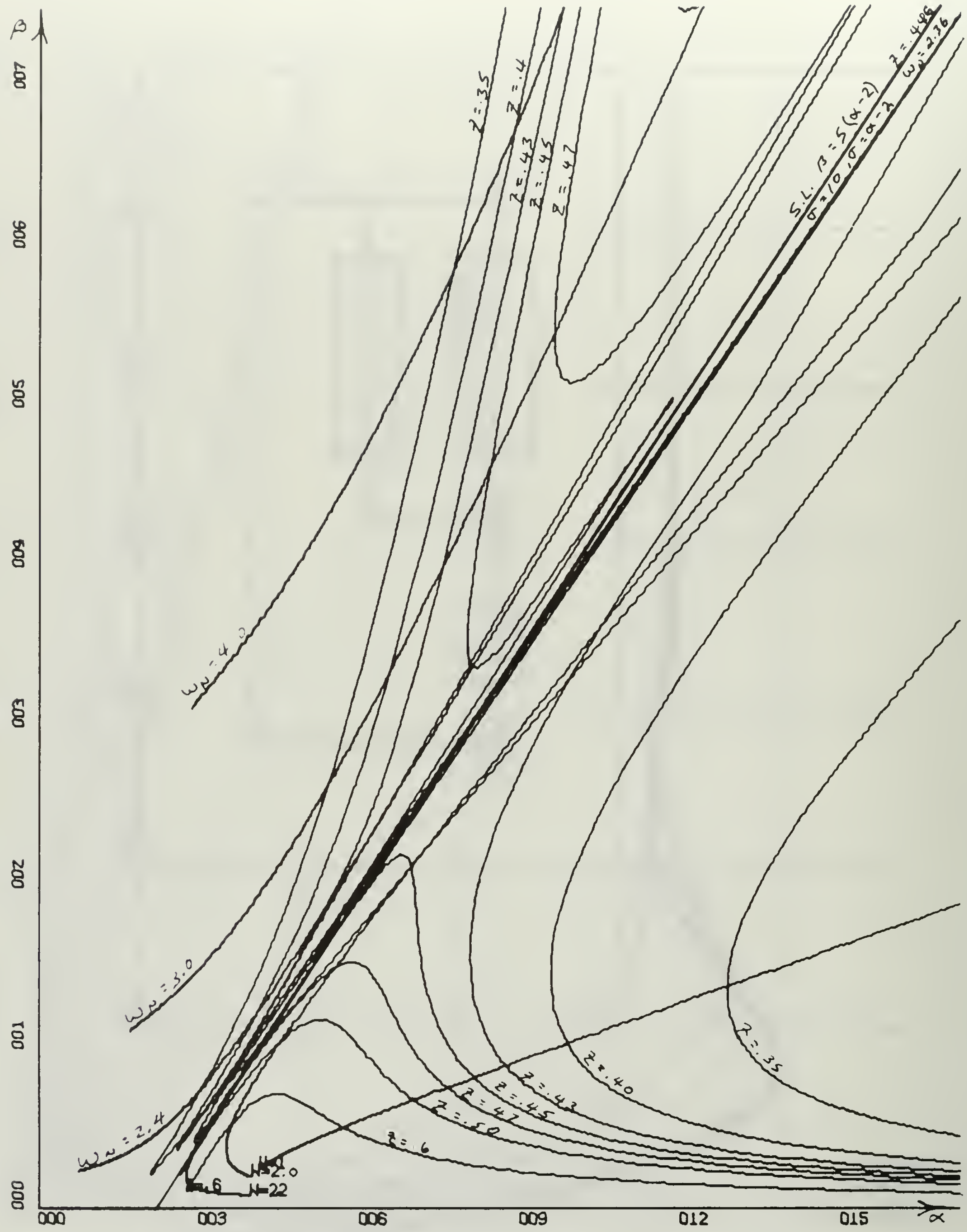
$$s^3+\alpha s^2+(4\beta+5)s+\beta=0$$

Figure IV-6 Parameter Plane Curves - Example 3



x scale = 1.0 units/in
y scale = .3 units/in

Figure IV-7 Step Response Curves - Example 4



x scale = 3 units/in
y scale = 10 units/in

$$s^4+(\alpha+10)s^3+(12\alpha+1)s^2+25\alpha s+10\beta=0$$

Figure IV-8 Parameter Plane Curves - Example 4

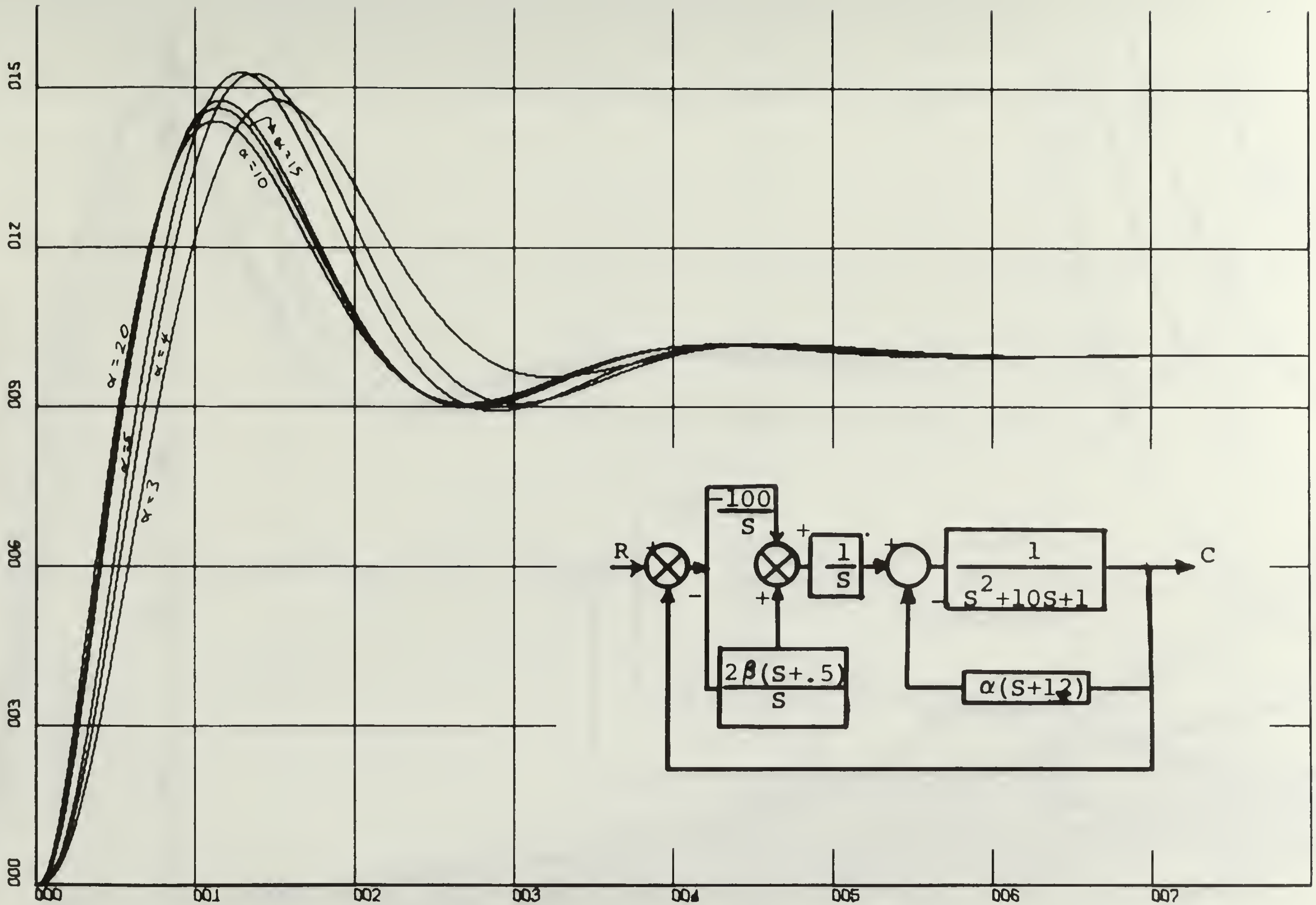


Figure IV-9 Step Response Curves - Example 5

x scale = 3 units/in
y scale = 10 units/in

$$s^4+(\alpha+10)s^3+(12\alpha+1)s^2+\beta s+2\beta-100=0$$

Figure IV-10 Parameter Plane Curves - Example 5

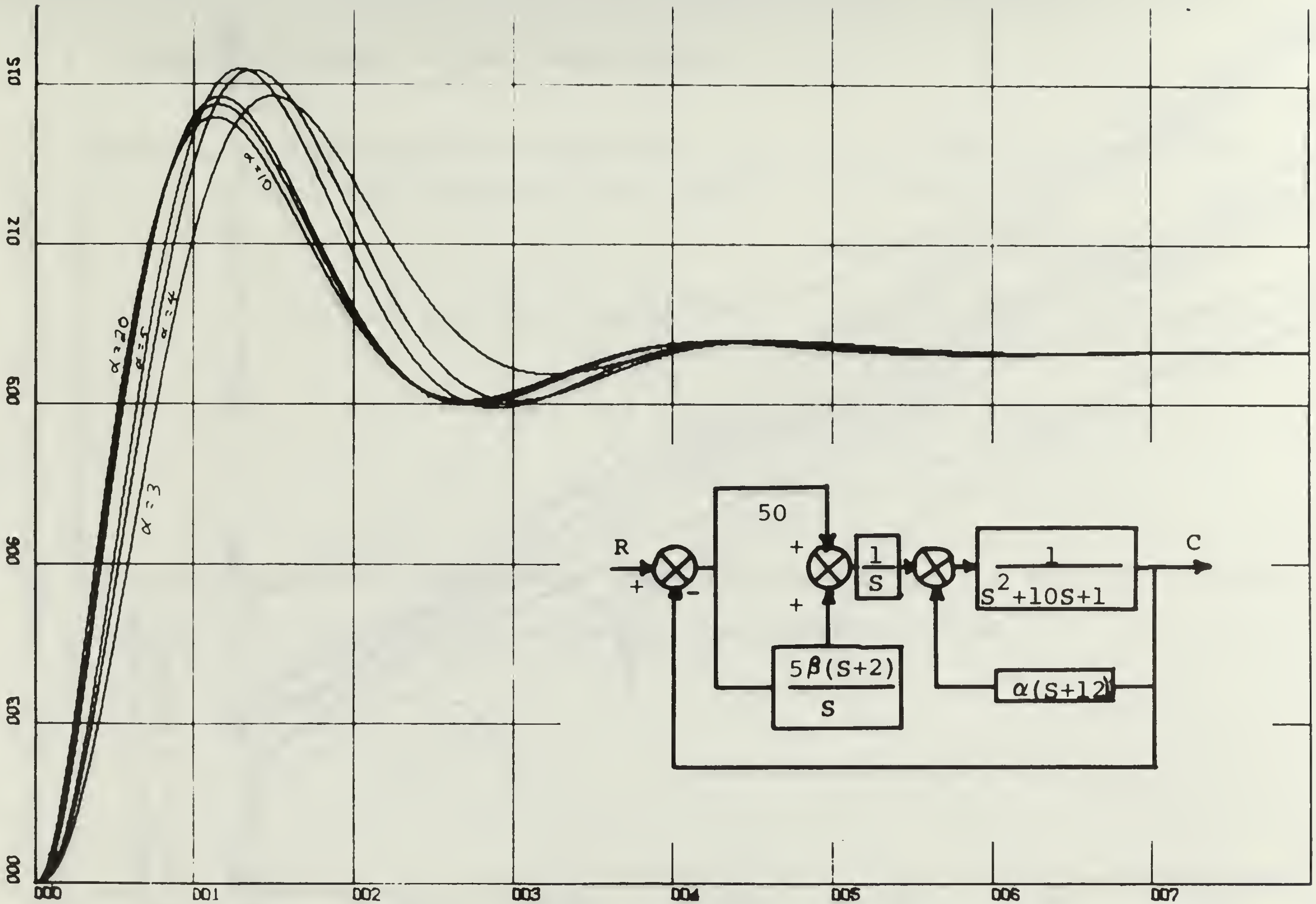
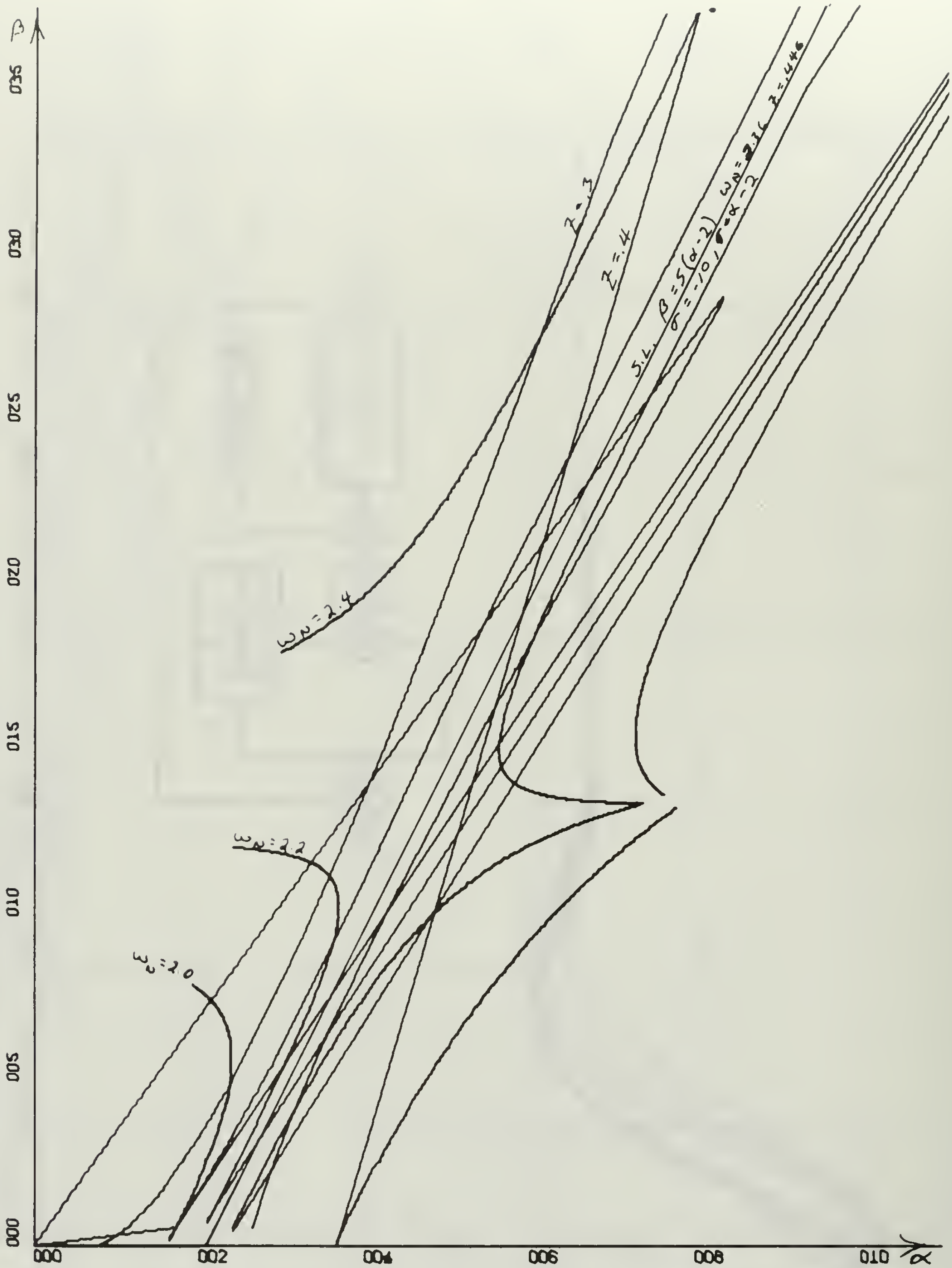


Figure IV-11 Step Response Curves - Example 6



x scale = 2 units/in
y scale = 5 units/in

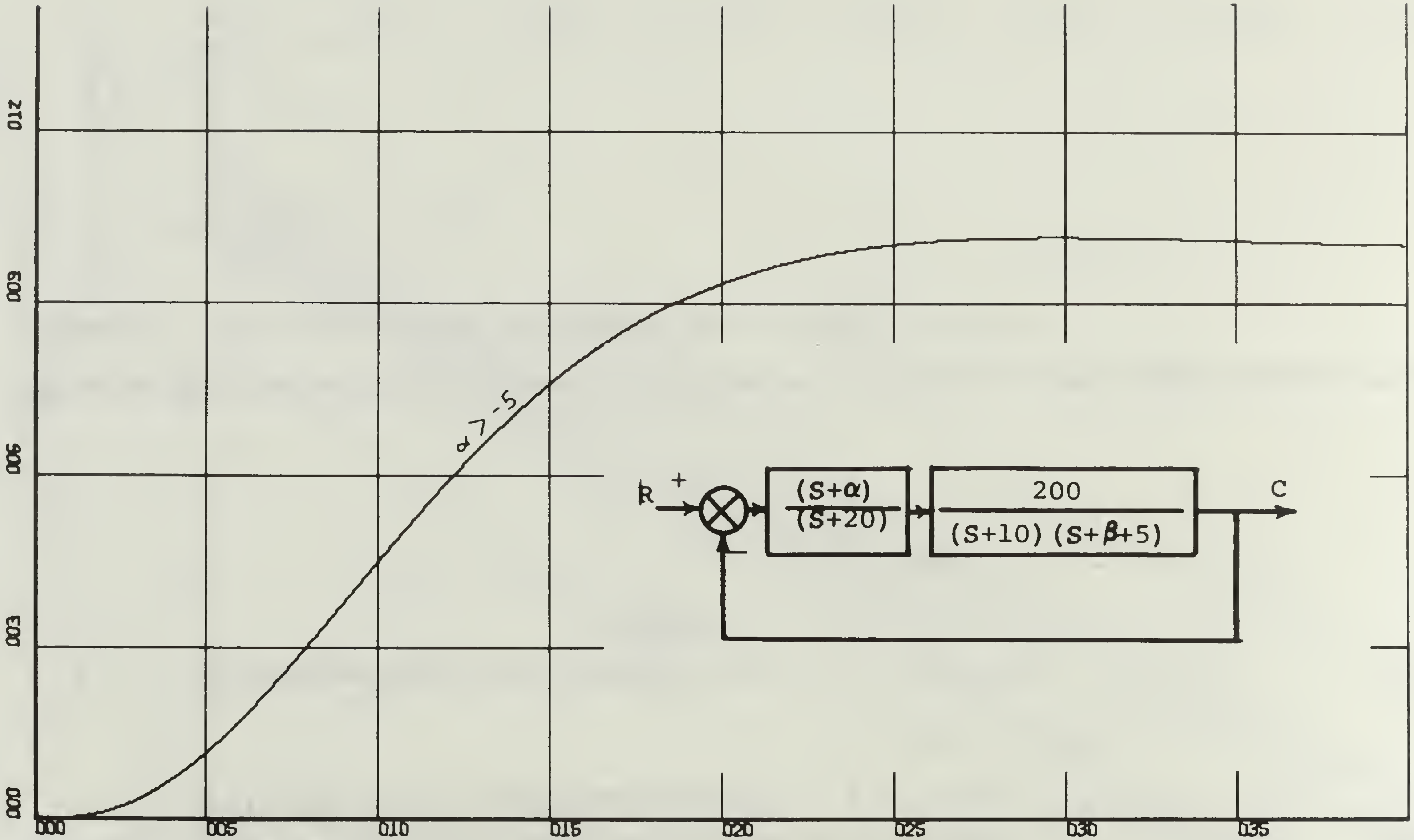
$$s^4+(\alpha+10)s^3+(12\alpha+1)s^2+(5\beta+50)s+10\beta=0$$

Figure IV-12 Parameter Plane Curves - Example 6

B. SYSTEMS WITH POLE - ZERO CANCELLATION

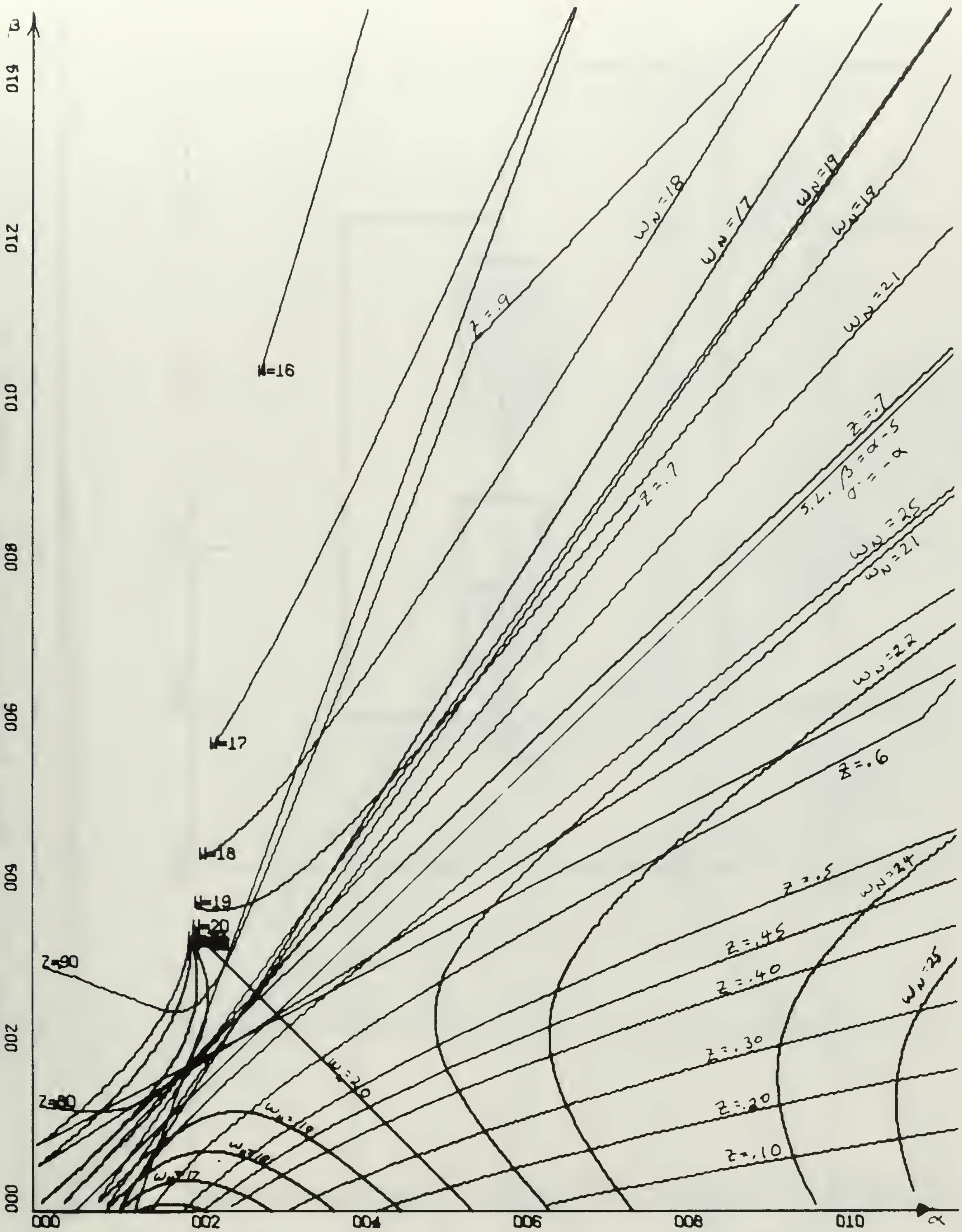
Example	Characteristic Equation	Roots	Singular Line	Applicable Figures
7	$s^3+(\beta+35)s^2+(30\beta+550)s+200(\alpha+\beta+5)=0$	$s = -(\beta+5)$	$\beta = \alpha-5$	IV-13
		$s = -15 \pm j13,22$		IV-14
8	$s^4+(\beta+40)s^3+40(\beta+10)s^2+400(\alpha+7.5)s$	$s = -5 \pm j\sqrt{75}$	$\beta = \alpha$	IV-15
	$+ 3000\alpha=0$	$s = -\alpha$		IV-16
		$s = -30$		

The pole-zero cancellation causes these systems to have the same step response for all values of α as shown by the applicable step response curves.



x scale = .05 units/in
y scale = .3 units/in

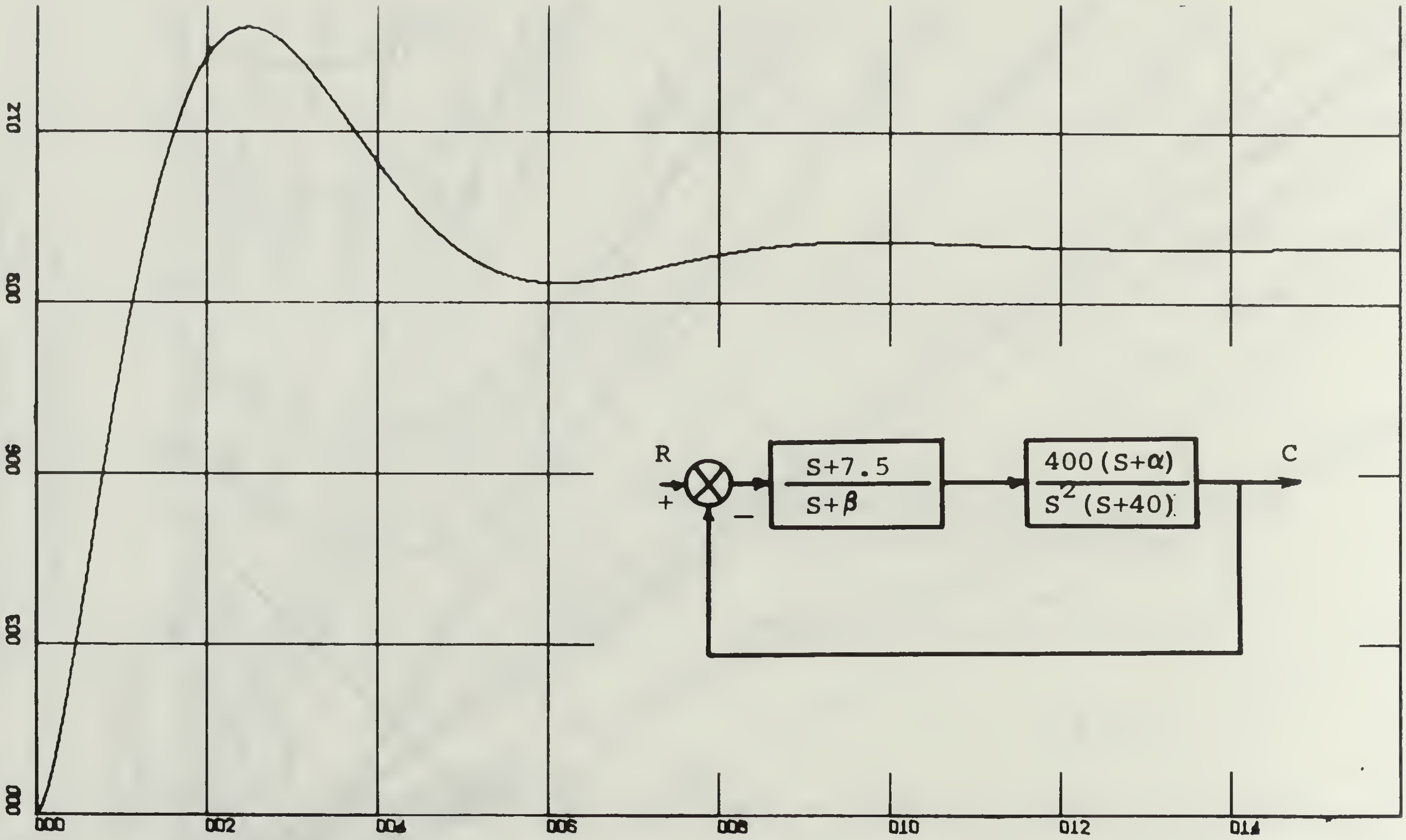
Figure IV-13 Step Response Curves - Example 7



x scale = 20 units/in
y scale = 20 units/in

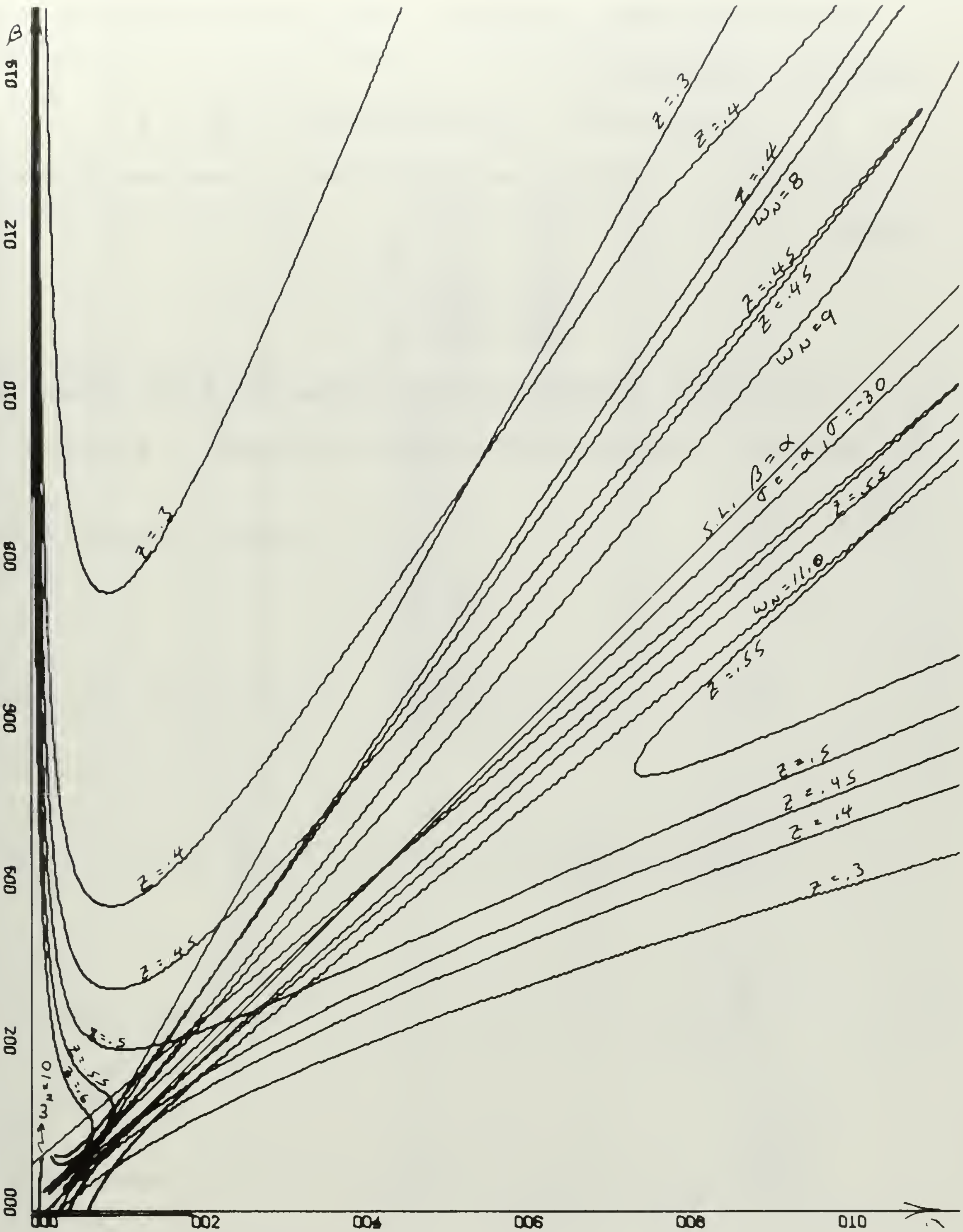
$$s^3+(\beta+35)s^2+(30\beta+550)s+200(\alpha+\beta+5)=0$$

Figure IV-14 Parameter Plane Curves - Example 7



x scale = .5 units/in
y scale = .3 units/in

Figure IV-15 Srep Response Curves - Example 8



x scale = 20 units/in
y scale = 20 units/in

$$s^4+(\beta+40)s^3+40(\beta+10)s^2+400(\alpha+7.5)s+3000\alpha=0$$

Figure IV-16 Parameter Plane Curves - Example 8

C. SYSTEM WITH LOCKED ROOTS

The system shown in Figure IV-17 has the following characteristic equation:

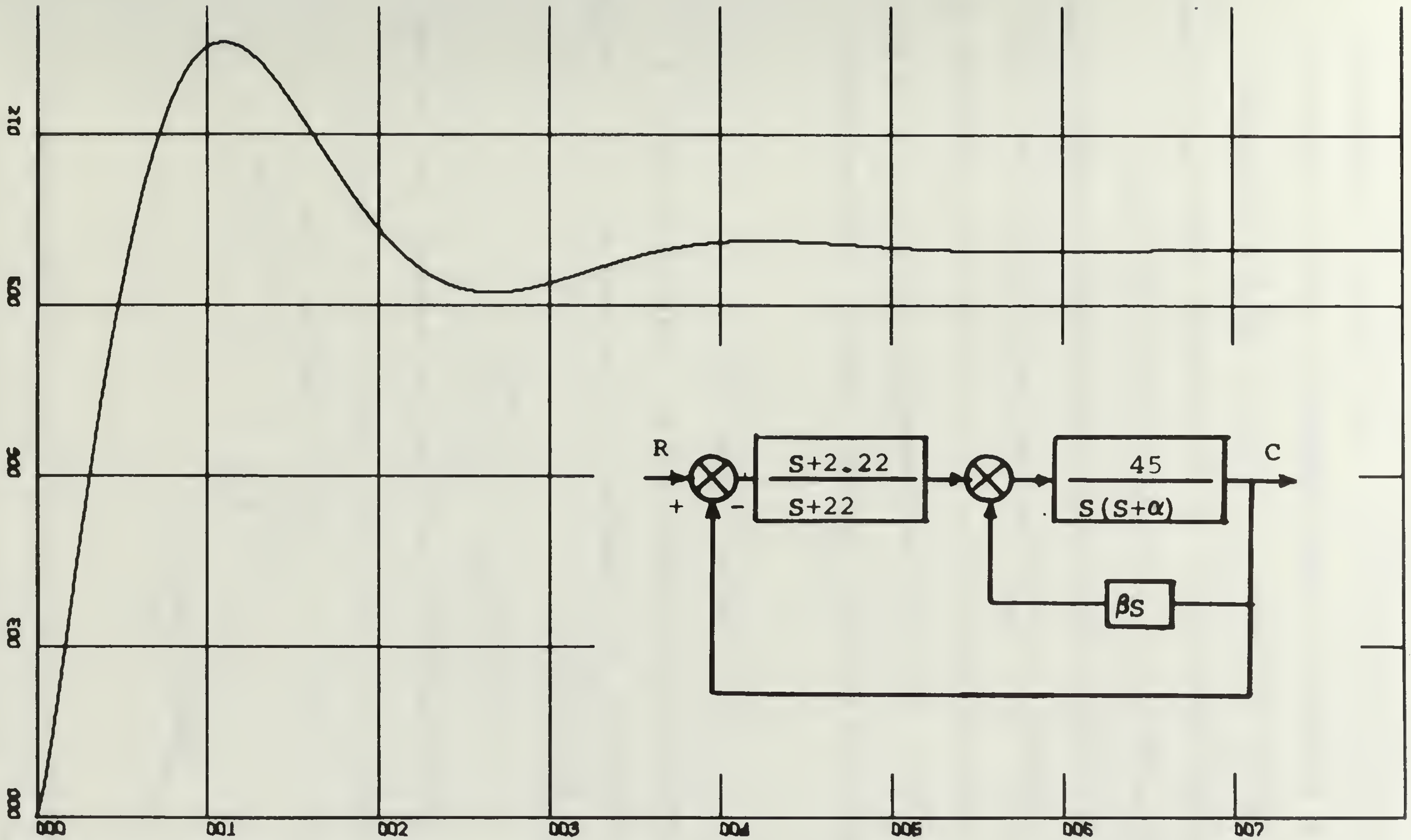
$$s^3 + (\alpha+45\beta+22)s^2 + (22\alpha+990\beta+45)s + 100 = 0 \ .$$

Setting $\alpha = -45\beta$ forces the system to have the following roots:

$$s = -20$$

$$s = -1 \pm j2 \quad .$$

There are no parameter plane curves for this system since the parameter consists of an infinite number of singular lines.



x scale = .2 units/in
y scale = .3 units/in

Figure IV-17 Step Response Curves - Example 9

V. SELF-ADAPTIVE CONTROL USING SINGULAR LINES

One important application of singular line systems is that of self-adaptive control. For example, a parameter which is a function of environmental conditions could be compensated for by linearly varying a second parameter. This situation might occur in an aircraft where the dynamic flight characteristics are a function of altitude.

Consider Figure V-1, below, which shows a singular line for $\zeta = \zeta_s$ and $\omega_n = \omega_{ns}$ on the parameter plane. β is the varying plane parameter and α is an adjustable parameter in the controller. The initial operating condition is on the singular line at point P_1 so that dominant complex roots are at

$$s = -\zeta_s\omega_{ns} \pm j\omega_{ns}\sqrt{1-\zeta_s^2} \ .$$

When β is varied by an amount $\Delta\beta$ the operating point is changed to point P_2 by changing α by an amount, $\Delta\alpha$. The system maintains roots at

$$s = -\zeta_s\omega_{ns} \pm j\omega_{ns}\sqrt{1-\zeta_s^2} \ .$$

The other characteristic roots may be changed, but for reasonable changes in α and β root dominance is retained.

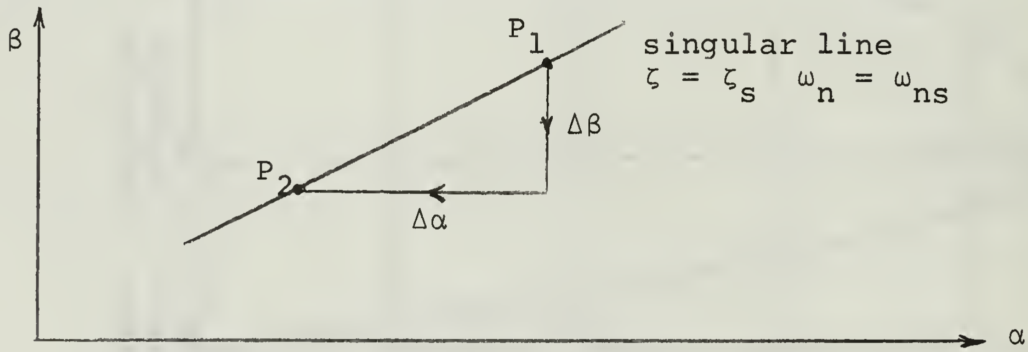


Figure V-1 Basic Concept of Self-Adaptive Control

The system in Figure V-2 is an example of a singular line plant (pole-zero cancellation) which has been designed for self-adaptive control by a procedure developed by R. A. Desrosiers (13). Explanation of the various control functions will not be explained but may be found in reference (13). Figure V-2 also shows the step response of the system with a 12% step change of β at t - .09 sec. As indicated, the adapted plant and model responses coincide. (The model is the basic plant with no parameter changes.)

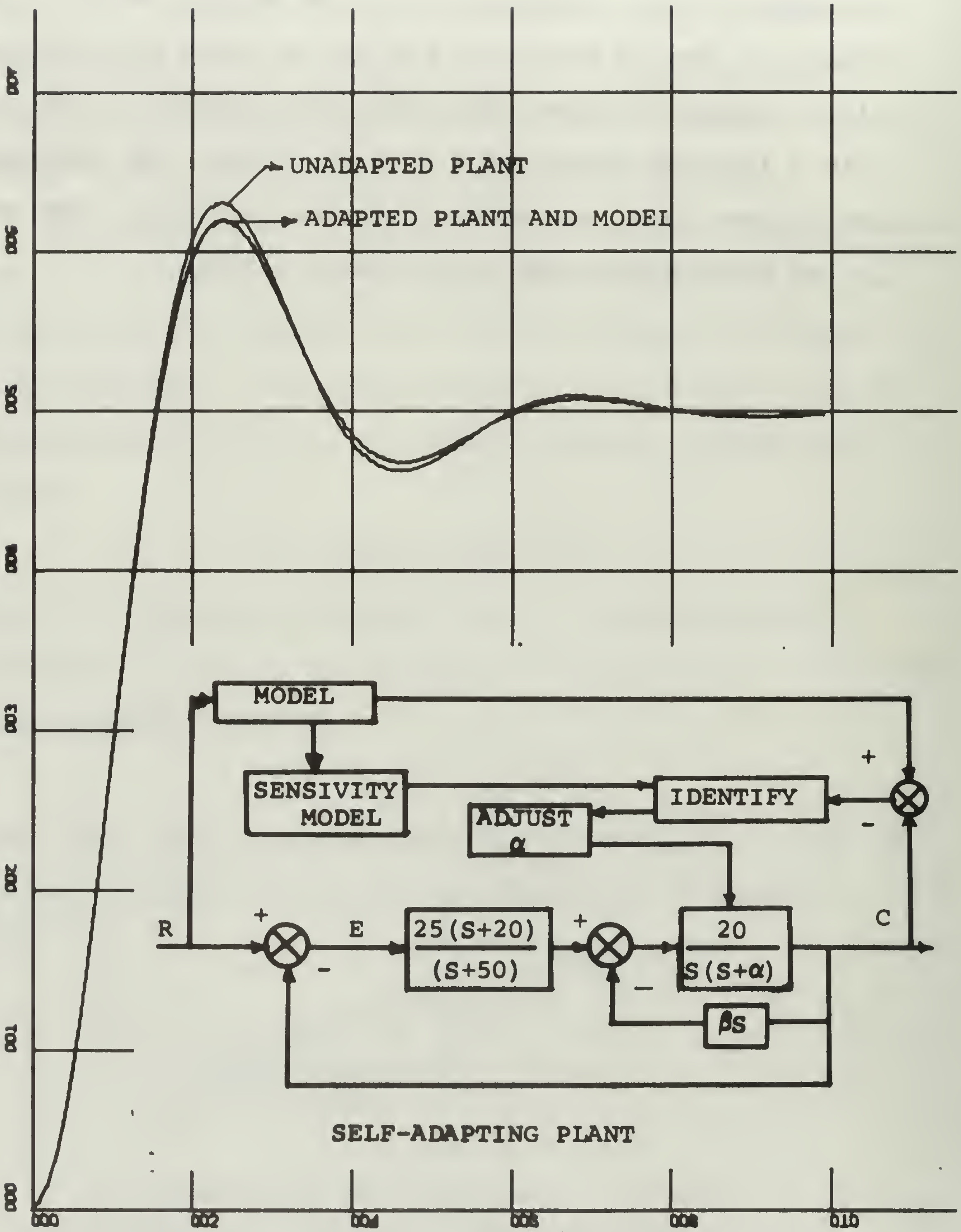


Figure V-2 Step Responses for Self-Adapting Plant

VI. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

Feedback control systems containing singular lines, while occurring in a limited number of cases, provide a tool for the design engineer working in self-adaptive control and other areas. Methods for predicting the existence of singular lines have been shown with examples given in the use of these methods. As indicated in previous sections of this report, the equations resulting from the application of known existence techniques are complex and do not, in general, provide a tractable means of finding singular line systems.

From the standpoint of the design engineer, it would be desirable to formulate a general set of rules which would provide a specific system with singular lines. Research to date does not indicate the existence of such a general set of rules. Further development is needed in this area.

A second approach might be to design a number of singular line systems using characteristic equations as starting points. A set of typical singular line systems would then be made available to the design engineer for correlation between a specific system and possible singular line characteristics.

Another area for future work might be to develop further theorems for singular lines on the parameter plane. For example, all real root lines for singular line systems were found to lie on the singular line. Further study may prove that this is true in general.

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APPENDIX I

TABLE OF THE CHEBYSHEV FUNCTIONS OF THE SECOND KIND

The Chebyshev functions of the second kind, $U_k(\zeta)$, are defined by the recursion relation:

$$U_{k+1}(\zeta) - 2\zeta U_k(\zeta) + U_{k-1}(\zeta) = 0$$

where:

$$U_0(\zeta) = 0$$

$$U_1(\zeta) = 1 \quad .$$

These functions for k = -1,..., 10 are

k	$U_k(\zeta)$
-1	-1
0	0
1	1
2	2ζ
3	$4\zeta^2 - 1$
4	$8\zeta^3 - 4\zeta$
5	$16\zeta^4 - 12\zeta^2 + 1$
6	$32\zeta^5 - 32\zeta^3 + 6\zeta$
7	$64\zeta^6 - 80\zeta^4 + 24\zeta^2 - 1$
8	$128\zeta^7 - 192\zeta^5 + 80\zeta^3 - 8\zeta$
9	$256\zeta^8 - 448\zeta^6 + 240\zeta^4 - 40\zeta^2 + 1$
10	$512\zeta^9 - 1024\zeta^7 + 672\zeta^5 - 160\zeta^3 + 10\zeta$

APPENDIX II

TABLE OF FUNCTIONS $U_k(\zeta)$ [3]

	U_{-1}	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
0.00				0.0	-1.00	0.000	1.0000	0.00000	-1.000000	0.0000000
0.05				0.1	-0.99	-0.199	0.9701	0.29601	-0.940499	-0.3900599
0.10				0.2	-0.96	-0.392	0.8816	0.56832	-0.767936	-0.7219072
0.15				0.3	-0.91	-0.573	0.7381	0.79443	-0.499771	-0.9443613
0.20				0.4	-0.84	-0.736	0.5456	0.95424	-0.163904	-1.0198016
0.25				0.5	-0.75	-0.875	0.3125	1.03125	0.203125	-0.9296875
0.30				0.6	-0.64	-0.984	0.0496	1.01376	0.558656	-0.6785664
0.35				0.7	-0.51	-1.057	-0.2299	0.89607	0.857149	-0.2960657
0.40				0.8	-0.36	-1.088	-0.5104	0.67968	1.054144	0.1636352
0.45				0.9	-0.19	-1.071	-0.7739	0.37449	1.110941	0.6253569
0.50	-1	0	1	1.0	0.00	-1.000	-1.000	0.00000	1.000000	1.0000000
0.55				1.1	0.21	-0.869	-1.1659	-0.41439	0.711061	1.1956571
0.60				1.2	0.44	-0.672	-1.2464	-0.82368	0.257984	1.1332608
0.65				1.3	0.69	-0.403	-1.2139	-1.17507	-0.313691	0.7672717
0.70				1.4	0.96	-0.056	-1.0384	-1.39776	-0.918464	0.1119104
0.75				1.5	1.25	0.375	-0.6875	-1.40625	-1.421875	-0.7265625
0.80				1.6	1.56	0.896	-0.1264	-1.09824	-1.630784	-1.5110144
0.85				1.7	1.89	1.513	0.6821	-0.35343	-1.282931	-1.8275527
0.90				1.8	2.24	2.232	1.7776	0.96768	-0.035776	-1.0320768
0.95				1.9	2.61	3.059	3.2021	3.02499	2.545381	1.9112339
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13. ABSTRACT

Derivation and methods for the proof of existence of singular lines on the parameter plane is presented. A method is derived for producing singular line systems. Several systems are designed and evaluated using this technique.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
PARAMETER PLANE						
SINGULAR LINES						

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